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Evasion with Weak Superiority*

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1. INTRODUCTION

We consider a differential game governed by the following

$$\begin{aligned} \dot{z} &= Az + f(u, v), \qquad t \geqslant 0 \\ z(0) &= z_0, \end{aligned} \tag{1.1}$$

where, z, $z_0 \in \mathbb{R}^n$, are the state and initial state, $u \in U \subseteq \mathbb{R}^p$, $v \in V \subseteq \mathbb{R}^q$ are the controls of the pursuer and the evader, respectively, A is an $(n \times n)$-matrix and $f\colon U \times V \to \mathbb{R}^n$ is a given mapping. Also, we are given a subspace M of $\mathbb{R}^n$. The game is ended when the state z reaches set M. Thus, M is called the terminal set. The goal of the pursuer is to terminate the game by choosing a suitable control

$$u(\cdot) \in \mathscr{U} \triangleq \{u\colon [0, \infty) \to U, \text{ measurable}\},$$

while the goal of the evader is to prevent the game from terminating by choosing a proper control

$$v(\cdot) \in \mathscr{V} \triangleq \{v\colon [0, \infty) \to V, \text{ measurable}\}.$$

In an evasion game, which is considered in this paper, the pursuer chooses his control $u(\cdot) \in \mathscr{U}$ at the start of the game. The evader, on the other hand, chooses the values of his control as the game evolves and can use $\{z(s), u(s) \mid 0 \leqslant s \leqslant t\}$ when he chooses the value $v(t)$ of the evasion control $v(\cdot) \in \mathscr{V}$ at time t. We denote $\hat{\mathscr{V}}$ to be the set of all such evasion controls.

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DEFINITION 2.1. The game is said to be evadable, if for any $z_0 \in \mathbb{R}^n \setminus M$, $u(\cdot) \in \mathcal{U}$, there exists a $v(\cdot) \in \mathcal{V}$, such that

$$d(M, z(t)) > 0, \qquad \forall t \geqslant 0.$$

where, $z(\cdot)$ is the trajectory of (1.1) corresponding to $u(\cdot)$ and $v(\cdot)$; $d(\cdot, \cdot)$ is the Euclidean distance in $\mathbb{R}^n$.

The evadability problem of differential evasion games has been extensively studied by many authors, see [1–7], for example, and the references cited therein. In the discussion of evadability, one usually assumed some superiority of the evader over the pursuer. In particular, for a game of form (1.1), one often essentially assumes, that

$$0 \in \operatorname{Int}\left(\bigcap_{u \in U} f(u, V)\right). \tag{1.2}$$

Namely, the evader has more than enough "power" to overcome the effect of the pursuit control. In [6], Satimov discusses a very special case in which the evader and the pursuer have the same "power." He obtained the evadability of the game by using the particular structure of the system.

The purpose of this paper is to use the idea of [6] to study our game (1.1). We will get the evadability of the game under very weak superiority of the evader to the pursuer.

2. PRELIMINARIES

Suppose A is an $(n \times n)$-matrix. Its minimal polynomial is the following

$$\varphi(\lambda) = \lambda^k - a_{k-1}\lambda^{k-1} - \cdots - a_0, \tag{2.1}$$

where $k \leqslant n$. Let us denote

$$\hat{A} = \begin{pmatrix} 0 & & & & & a_0 \\ 1 & 0 & & & & a_1 \\ & 1 & \ddots & & & a_2 \\ & & \ddots & \ddots & & \vdots \\ & & & \ddots & 0 & a_{k-2} \\ & & & & 1 & a_{k-1} \end{pmatrix}_{k \times k}, \qquad a = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{k-2} \\ a_{k-1} \end{pmatrix}_{k \times 1}.$$

Also, we let $e_i = (0, ..., 0, 1^{(i)}, 0, ..., 0)^T \in \mathbb{R}^k$, $0 \leqslant i \leqslant k-1$.

LEMMA 2.1. *For all* $t \in \mathbb{R}$,

$$e^{At} = \sum_{i=0}^{k-1} g_i(t) A^i, \tag{2.2}$$

where

$$g_i(t) = \frac{t^i}{i!} + \frac{1}{k!} \int_0^t (t-\tau)^k \, (e_i, e^{\hat{A}\tau} a) \, d\tau, \qquad 0 \leqslant i \leqslant k-1. \tag{2.3}$$

Proof. Let $g(t) \equiv (g_0(t), ..., g_{k-1}(t))^T$ be the unique solution of the following problem

$$\begin{aligned} \frac{dg}{dt} &= \hat{A} g \\ g(0) &= e_0. \end{aligned} \tag{2.4}$$

Then,

$$\begin{aligned} g^{(i)}(0) &= \hat{A}^i e_0 = e_i, \qquad 0 \leqslant i \leqslant k-1 \\ g^{(k)}(t) &= \hat{A}^k g(t) = e^{\hat{A}t} \hat{A}^k e_0 = e^{\hat{A}t} a. \end{aligned}$$

Hence, by Taylor expansion,

$$\begin{aligned} g_i(t) &= \frac{t^i}{i!} + \frac{1}{k!} \int_0^t (t-\tau)^k \, g_i^{(k)}(\tau) \, d\tau \\ &= \frac{t^i}{i!} + \frac{1}{k!} \int_0^t (t-\tau)^k \, (e_i, e^{\hat{A}\tau} a) \, d\tau, \qquad 0 \leqslant i \leqslant k-1. \end{aligned}$$

Now, let us set

$$F(t) = e^{At} - \sum_{i=0}^{k-1} g_i(t) \, A^i.$$

Then, one has, noting $\varphi(A) = 0$, that

$$\begin{aligned} F'(t) &= Az^{At} - \sum_{i=0}^{k-1} g_i'(t) \, A^i \\ &= Ae^{At} - \left\{ a_0 g_{k-1}(t) \, I + \sum_{i=1}^{k-1} (g_{i-1}(t) + a_i g_{k-1}(t)) \, A^i \right\} \\ &= Ae^{At} - \left\{ \sum_{i=1}^{k-1} g_{i-1}(t) \, A^i + g_{k-1}(t)[A^k - \varphi(A)] \right\} \\ &= A \left(e^{At} - \sum_{i=0}^{k-1} g_i(t) \, A^i \right) \equiv AF(t). \end{aligned}$$

Since $F'(0) = 0$, our conclusion follows. ∎

Remark 2.2. By the uniqueness of the solution of (2.4), one has

$$g(t) \neq 0, \qquad \forall t \in \mathbb{R}. \tag{2.5}$$

Also, from (2.3), we have the existence of a $\theta \in (0, 1]$, such that

$$g_i(t) > 0, \qquad g_i'(t) > 0, \qquad t \in (0, \theta], \qquad 0 \leqslant i \leqslant k-1. \tag{2.6}$$

LEMMA 2.3. *Let $p_1(t), \ldots, p_m(t)$ be measurable, bounded functions defined on $[0, T]$, with values in $\mathbb{R}^r$. Let $\mu_1(t), \ldots, \mu_m(t)$ be measurable nonnegative scalar functions defined on $[0, T]$, satisfying $\sum_{i=1}^m \mu_i(t) = 1$. Then, for any $\varepsilon > 0$, there exists a measurable function $p(\cdot)$ with values $p(t)$ in the set $\{p_1(t), \ldots, p_m(t)\}$ at any $t \in [0, T]$, and the value $p(t)$ of $p(\cdot)$ at time t only depends on $\{\mu_i(s), p_i(s) | 0 \leqslant s \leqslant t, 1 \leqslant i \leqslant m\}$, such that for any nonnegative nondecreasing scalar function $q(\cdot)$,*

$$\sup_{t \in [0, T]} \left\| \int_0^t q(t-\tau) \left[\sum_{i=1}^m \mu_i(\tau) p_i(\tau) - p(\tau) \right] d\tau \right\| \leqslant \sqrt{r}\, q(T)\, \varepsilon. \tag{2.7}$$

Proof. By Lemma 2.1 of [2], we know that for any $\varepsilon > 0$, there exists a measurable function $p(\cdot)$ satisfying all the requirements we want except (2.7) and such that

$$\sup_{t \in [0, T]} \left\| \int_0^t \left[\sum_{i=1}^m \mu_i(\tau) p_i(\tau) - p(\tau) \right] d\tau \right\| < \varepsilon.$$

Then, let us denote $p(\cdot) = (p^1(\cdot), \ldots, p^r(\cdot))^T$ and $p_i(\cdot) = (p_i^1(\cdot), \ldots, p_i^r(\cdot))^T$. By the Second Mean Value Theorem, we have, for $t \in [0, T]$, that

$$\begin{aligned} &\left\| \int_0^t q(t-\tau) \left[\sum_{i=1}^m \mu_i(\tau) p_i(\tau) - p(\tau) \right] d\tau \right\|^2 \\ &\quad = \sum_{j=1}^r \left| \int_0^t q(t-\tau) \left[\sum_{i=1}^m \mu_i(\tau) p_i^j(\tau) - p^j(\tau) \right] d\tau \right|^2 \\ &\quad = \sum_{j=1}^r \left| q(t) \int_0^{\xi_j} \left[\sum_{i=1}^m \mu_i(\tau) p_i^j(\tau) - p^j(\tau) \right] d\tau \right|^2 \\ &\quad < r q(t)^2 \varepsilon^2. \quad \blacksquare \end{aligned}$$

3. EVADABILITY

In this section, we prove the main result of this paper.

THEOREM 3.1. *In the evasion game defined by* (1.1) *and terminal set M let the following hold*:

(1) *The function f is continuous.*

(2) *The sets U and V are compact.*

(3) *The minimal polynomial of A is given by* (2.1) *with $k<n$.*

(4) *The terminal set M is a subspace of $\mathbb{R}^n$ with* $\dim M^\perp = m \geqslant k$

(5) *There exist $0 \leqslant i_0 \leqslant k-1$ and $\delta > 0$, such that*

$$\Pi A^i f(U, V) = \{0\}, \qquad 0 \leqslant i \leqslant i_0 - 1 \tag{3.1}$$

$$0 \in \bigcap_{u \in U} co \begin{pmatrix} \Pi A^{i_0} f(u, V) \\ \vdots \\ \Pi A^{k-1} f(u, V) \end{pmatrix} \tag{3.2}$$

$$\inf_{u \in U} \inf_{\psi \in M^\perp, \|\psi\| = 1} \max_{v \in V} |(\psi, \Pi A^{i_0} f(u, v))| \geqslant \delta, \tag{3.3}$$

where, $\Pi \colon \mathbb{R}^n \to M^\perp$ is the orthogonal projection. Then, the game (1.1) *with terminal set M is evadable.*

Remark 3.2. It is clear that (3.2) is implied by

$$0 \in \bigcap_{u \in U} co\, f(u, V). \tag{3.4}$$

Proof of Theorem 3.1. Let $z_0 \in \mathbb{R}^n \backslash M$, $u(\cdot) \in \mathscr{U}$ be given. Let $0 < \theta \leqslant 1$ be as in (2.6). Also, we have $L > 0$, such that

$$\begin{aligned} &|g_i(t)| \leqslant L, && t \in [0, \theta], 0 \leqslant i \leqslant k-1 \\ &\|\Pi A^i f(u, v)\| \leqslant L, && (u, v) \in U \times V, 0 \leqslant i \leqslant k-1 \\ &|(e_i, e^{\hat{A}t} a)| \leqslant L, && t \in [0, \theta], 0 \leqslant i \leqslant k-1. \end{aligned} \tag{3.5}$$

Case 1. Suppose for $t \in [0, \theta]$,

$$\Pi e^{At} z_0 \neq 0. \tag{3.6}$$

Then, we let

$$b = \inf_{0 \leqslant t \leqslant \theta} \|\Pi e^{At} z_0\| > 0.$$

By (3.2), for given $u(\cdot) \in \mathscr{U}$ we have, using Filippov's lemma, $v_j(\cdot) \in \mathscr{V}$ and $\mu_j(\cdot)$, $1 \leqslant j \leqslant r \equiv (k - i_0) m + 1$, such that

$$\sum_{j=1}^r \mu_j(t) = 1, \qquad \mu_j(t) \geqslant 0, \qquad 1 \leqslant j \leqslant r$$

$$\sum_{j=1}^r \mu_j(t) \begin{pmatrix} \Pi A^{i_0} f(u(t), & v_j(t)) \\ & \vdots \\ \Pi A^{k-1} f(u(t), & v_j(t)) \end{pmatrix} = 0.$$

Then, we take

$$\varepsilon = \frac{b}{2(k-i_0)\,L} > 0.$$

By Lemma 2.3, there exists a measurable function $p(\cdot)$, such that $p(t)$ only depends on $\{\mu_j(s),\ \Pi A^i f(u(s), v_j(s)) \mid 0 \leqslant s \leqslant t,\ 1 \leqslant j \leqslant r,\ i_0 \leqslant i \leqslant k-1\}$, and

$$p(t) \in \left\{ \begin{pmatrix} \Pi A^{i_0} f(u(t), v_j(t)) \\ \vdots \\ \Pi A^{k-1} f(u(t), v_j(t)) \end{pmatrix} \middle| \, 1 \leqslant j \leqslant r \right\}$$

$$\sup_{t \in [0,\theta]} \left\| \int_0^t q(t-\tau)\, p(\tau)\, d\tau \right\| \leqslant \sqrt{r}\, q(\theta)\, \varepsilon$$

for any nonnegative nondecreasing scalar function $q(\cdot)$. It is clear that there exists $\hat{v}(\cdot) \in \hat{\mathscr{V}}$, such that

$$p(t) = \begin{pmatrix} \Pi A^{i_0} f(u(t), \hat{v}(t)) \\ \vdots \\ \Pi A^{k-1} f(u(t), \hat{v}(t)) \end{pmatrix}, \qquad 0 \leqslant t \leqslant \theta.$$

Thus, by taking the evasion control $\hat{v}(\cdot)$, one has

$$\begin{aligned} d(M, z(t)) &= \|\Pi z(t)\| \\ &\geqslant \|\Pi e^{At} z_0\| - \left\| \sum_{i=0}^{k-1} \int_0^t \Pi A^i f(u(\tau), \hat{v}(\tau))\, g_i(t-\tau)\, d\tau \right\| \\ &\geqslant b - \sum_{i=i_0}^{k-1} \left\| \int_0^t g_i(t-\tau)\, p(\tau)\, d\tau \right\| \\ &\geqslant b - \sum_{i=i_0}^{k-1} \sqrt{r}\, g_i(\theta)\, \varepsilon \\ &\geqslant b - (k-i_0)\sqrt{r}\, L\varepsilon = \frac{b}{2}, \qquad t \in [0, \theta]. \end{aligned}$$

Now, at time $t = \theta$, if we have

$$\Pi e^{At} z(\theta) \neq 0, \qquad 0 \leqslant t \leqslant \theta,$$

then, we can repeat the above argument to get the evadability on $[\theta, 2\theta]$ and so on. Otherwise, we are in the following situation.

Case 2. Suppose for some $t_0 > 0$,

$$\Pi e^{At_0} z_0 \equiv \sum_{i=0}^{k-1} g_i(t_0)\, \Pi A^i z_0 = 0. \tag{3.7}$$

By (2.5), we get that $\{\Pi A^i z_0 \mid 0 \leqslant i \leqslant k-1\}$ is linearly dependent. Hence,

$$\dim(\operatorname{Span}\{\Pi A^i z_0 \mid 0 \leqslant i \leqslant k-1\}) < k \leqslant m \equiv \dim M^{\perp}.$$

Thus, noting that $\Pi A^i z_0 \in M^{\perp}$, $0 \leqslant i \leqslant k-1$, we have the existence of a $\psi \in M^{\perp}$, $\|\psi\| = 1$, such that

$$(\psi, \Pi A^i z_0) = 0, \qquad 0 \leqslant i \leqslant k-1.$$

Now, by (3.3), for given $u(\cdot) \in \mathscr{U}$, we can take $\hat{v}(\cdot) \in \hat{\mathscr{V}}$, such that

$$|(\psi, \Pi A^{i_0} f(u(t), \hat{v}(t)))| \geqslant \delta.$$

Then, under this evasion control, we have, for $t \in [0, \theta]$, that

$$\begin{aligned} &d(M, z(t)) \\ &\quad = \|\Pi z(t)\| \geqslant |(\psi, \Pi z(t))| \\ &\quad = \left| \sum_{i=0}^{k-1} (\psi, \Pi A^i z_0)\, g_i(t) + \sum_{i=0}^{k-1} \int_0^t g_i(t-\tau)(\psi, \Pi A^i f(u(t), \hat{v}(t)))\, d\tau \right| \\ &\quad \geqslant \left| \sum_{i=i_0}^{k-1} \int_0^t g_i(t-\tau)(\psi, \Pi A^i f(u(t), \hat{v}(t)))\, d\tau \right| \\ &\quad \geqslant \int_0^t \delta\, g_{i_0}(\tau)\, d\tau - L \sum_{i=i_0+1}^{k-1} \int_0^t g_i(\tau)\, d\tau \\ &\quad \geqslant \delta \frac{t^{i_0+1}}{(i_0+1)!} - L \frac{t^{k+1}}{(k+1)!} - L \sum_{i=i_0+1}^{k-1} \left(\frac{t^{i+1}}{(i+1)!} + L \frac{t^{k+1}}{(k+1)!} \right) \\ &\quad \geqslant \frac{t^{i_0+1}}{(i_0+1)!} [\delta - Kt], \end{aligned}$$

where,

$$K = L \sum_{i=i_0+1}^{k} \frac{(i_0+1)!}{(i+1)!} \theta^{i-i_0-1} + L^2(k-i_0-1) \frac{(i_0+1)!}{(k+1)!} \theta^{k-i_0-1}.$$

Then, if we let $\theta_0 = \min\{\theta, \delta/2K\}$, we have

$$d(M, z(t)) \geqslant \frac{t^{i_0+1}}{(i_0+1)!} \frac{\delta}{2}, \qquad t \in (0, \theta_0].$$

Since $z(0) = z_0 \in \mathbb{R}^n \setminus M$, we get

$$d(M, z(t)) > 0, \qquad t \in [0, \theta_0].$$

Since θ and θ_0 are absolute constants, we can easily repeat the above arguments to get the evadability of the game. ∎

4. An Application

We present an example in this section. Let us consider a differential game foverned by the following system (cf. [3 or 5])

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ f_3(u, v) \\ f_4(u, v) \end{pmatrix} \equiv Az + f(u, v), \qquad (4.1)$$

where, $u = (u_1, u_2, u_3)^T$, $v = (v_1, v_2, v_3)^T$,

$$\begin{pmatrix} f_3(u, v) \\ f_4(u, v) \end{pmatrix} = \begin{pmatrix} \cos v_3 & -\sin v_3 \\ \sin v_3 & \cos v_3 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} - \begin{pmatrix} \cos u_3 & -\sin u_3 \\ \sin u_3 & \cos u_3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \qquad (4.2)$$

and $z_i \in \mathbb{R}$, $u_1^2 + u_2^2 \leqslant \lambda_1^2$, $v_1^2 + v_2^2 \leqslant \lambda_2^2$, $|u_3| \leqslant \beta_1$, $|v_3| \leqslant \beta_2$. The terminal set is $M = \{z_1 = z_2 = 0\}$.

In [5], some conditions were imposed on λ_1, λ_2, β_1, β_2 in order to get the evadability of the game. Here, we assume that $\lambda_1 = \lambda_2 \equiv \lambda > 0$, $\beta_1 = \beta_2 \equiv \beta \geqslant 0$, i.e., the evader and the pursuer have the same "power." It is clear that this case is not included in the cases discussed in [5]. In fact, we will see that the conditions of the main theorem of [5], which look similar to (3.3) are not satisfied in our case, in general.

It is clear that $A^2 = 0$. Thus, $k = 2 = \dim M^\perp$. Hence, conditions (1) and (2) of Theorem 3.1 hold. Also, if we let $\Pi: \mathbb{R}^4 \to M^\perp$ be the orthogonal projection, then

$$\Pi f(U, V) = \{0\} \qquad (4.3)$$

$$\Pi A f(u, v) = \begin{pmatrix} f_3(u, v) \\ f_4(u, v) \\ 0 \\ 0 \end{pmatrix}, \qquad (u, v) \in U \times V, \qquad (4.4)$$

where, $U = V = B_\lambda^2(0) \times [-\beta, \beta]$, $B_\lambda^2(0) \triangleq \{y \in \mathbb{R}^2 \mid \|y\| \leqslant \lambda\}$. Since we have

$$f(u, u) = 0, \qquad \forall u \in U, \qquad (4.5)$$

(3.4) holds, and hence so does (3.2). Now, we show that (3.3) holds. To this end, let $\psi \in M^\perp$, $\|\psi\| = 1$. Then, $\psi = (\eta^T, 0)^T$ with $\eta \in \mathbb{R}^2$, $\|\eta\| = 1$. Then for any $u \in U$, by taking $v \in V$ with $v_3 = u_3$, one has

$$(\psi, \Pi A f(u, v)) = \eta^T \begin{pmatrix} \cos u_3 & -\sin u_3 \\ \sin u_3 & \cos u_3 \end{pmatrix} \begin{pmatrix} u_1 - v_1 \\ u_2 - v_2 \end{pmatrix}.$$

Let us denote

$$\xi \equiv \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \triangleq \begin{pmatrix} \cos u_3 & \sin u_3 \\ -\sin u_3 & \cos u_3 \end{pmatrix} \eta.$$

Then, $\|\xi\| = 1$. Now, if we take

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = -\lambda[\operatorname{sgn}(u_1\xi_1 + u_2\xi_2)]\, \xi, \qquad (\operatorname{sgn} 0 \triangleq 1)$$

then

$$|(\psi, \Pi A f(u, v))| = \lambda + |u_1\xi_1 + u_2\xi_2| \geqslant \lambda. \tag{4.6}$$

Hence, (3.3) holds and the game is evadable.

We note from (4.2) that for $\beta \leqslant \pi/2$,

$$\bigcap_{u \in U} \Pi A f(u, V) = \{0\}. \tag{4.7}$$

Thus, in this case, the conditions of the main theorem of [5] are not satisfied.

Also, we can see that in order to have (3.4), we only need to take

$$V = \{(v_1, v_2, v_3) \mid v_1^2 + v_2^2 = \lambda^2,\ |v_3| \leqslant \beta\}.$$

It is clear that for this V, we still have (4.6). Thus the game is still evadable. But now, the evader is even "less" powerful than the pursuer.

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